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Comment on Asymptotic Properties of Coupled Langevin Equations

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The asymptotic behavior of coupled Langevin equations in the limit of weak noise is studied by general normal form techniques, in the vicinity of a pitchfork bifurcation. The non-Gaussian behavior of the critical variable is established. The conditional probability of the noncritical variable around the center manifold is determined. It is shown that in certain cases the distribution of this later variable may be non-Gaussian.

KEY WORDS: Stochastic processes; Langevin equation; nonlinear dynamics; normal forms; adiabatic elimination; center manifold.

In Ref. 1 the asymptotic behavior of coupled Langevin equations undergoing a cusp bifurcation was studied in the limit of weak noise. If $\mathbf{U} = (U_1, ..., U_N)$ are the variables, the equations are of the form

$$\partial_t \mathbf{U} = L\mathbf{U} + \mathbf{N}(\mathbf{U}) + \varepsilon^{1/2}\mathbf{F}(t; \mathbf{U}) \tag{1}$$

where $\mathbf{N}(\mathbf{U})$ are nonlinear deterministic terms, $\mathbf{F}(t; \mathbf{U})$ stands for the additive and multiplicative noise (ε measures the intensity of the noise), and the $N \times N$ matrix L is in diagonal form $L_{\alpha\beta} = \gamma_\alpha \delta_{\alpha\beta}$, $\gamma_1 = 0$, $\gamma_\alpha < 0$ for $\alpha \geqslant 2$. It was then shown working with the Fokker–Planck equation associated to (1) and using scaled variables to implement a singular perturbation technique that the critical variable U_1 has non-Gaussian fluctuations, while the variables U_α (which are linearly the fast variables)

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exhibit Gaussian or non-Gaussian behavior, depending on the nonlinearities in (1), which were classified accordingly.

The purpose of this comment is twofold. First, we want to show that the results concerning the behavior of $\mathbf{U}$ can be obtained by general normal form techniques[2,3] in which one deals directly with the Langevin equations. The method avoids the use of scalings in the intermediate steps and gives as a result the appropriate scaled variables which appear in the passage to the weak noise limit ($\varepsilon \to 0$). As a second point we discuss the different behaviors of the variables U_α, $\alpha \geqslant 2$, and we show that they can be understood as generic and nongeneric (i.e., of higher codimension) situations arising in the problem. Indeed, in the generic situation the variables U_α, $\alpha \geqslant 2$, exhibit a Gaussian scaling, while among the nongeneric cases of codimension 2 one can find in one case a non-Gaussian scaling for a fast variable U_α, $\alpha \geqslant 2$, which corresponds to a non-Gaussian behavior of this variable.

We consider then the stochastic differential equations (1) with $\mathbf{F}(t; \mathbf{U})$ of the form

$$\mathbf{F}(t; \mathbf{U}) = L^{(1)}(t)\mathbf{U} + \mathbf{M}(t; \mathbf{U}) + \mathbf{D}(t) \tag{2}$$

where

$$\mathbf{U} = \sum_{\alpha=1}^{N} U_\alpha \mathbf{e}_\alpha; \qquad \mathbf{D}(t) = \sum_{\alpha=1}^{N} D_\alpha(t)\, \mathbf{e}_\alpha$$

$$\mathbf{e}_1 = (1, 0,..., 0), \quad \mathbf{e}_2 = (0, 1, 0,..., 0),... \quad \mathbf{e}_N = (0,..., 0, 1)$$

$L^{(1)}(t)$ is an $N \times N$ matrix

$$L^{(1)}(t)\, \mathbf{e}_\alpha = \sum_{\beta=1}^{N} L^{(1)}_{\beta\alpha}(t)\, \mathbf{e}_\beta$$

and

$$\mathbf{N}(\mathbf{U}) = \sum_{r \geqslant 2} \mathbf{N}^{(r)}(\mathbf{U}), \qquad \mathbf{M}(t; \mathbf{U}) = \sum_{r \geqslant 2} \mathbf{M}^{(r)}(t; \mathbf{U})$$

with

$$\begin{aligned} \mathbf{N}^{(r)}(\mathbf{U}) &= \sum u^{(r)}_{\alpha;\alpha_1 \cdots \alpha_r} U_{\alpha_1} \cdots U_{\alpha_r} \mathbf{e}_\alpha \\ \mathbf{M}^{(r)}(t; \mathbf{U}) &= \sum v^{(r)}_{\alpha;\alpha_1 \cdots \alpha_r}(t)\, U_{\alpha_1} \cdots U_{\alpha_r} \mathbf{e}_\alpha \end{aligned} \tag{3}$$

We have $L\mathbf{e}_\alpha = \gamma_\alpha \mathbf{e}_\alpha$ and we assume nonresonant conditions between the eigenvalues γ_α. The matrix elements $L^{(1)}_{\alpha\beta}(t)$, the $v^{(r)}_{\alpha;\alpha_1 \cdots \alpha_r}(t)$, and $D_\alpha(t)$ are

Gaussian white noises with known correlations; in particular, the additive noise $\mathbf{D}(t)$ has correlations

$$\{D_\alpha(t)\, D_\beta(t')\} = Q_{\alpha\beta}\delta(t-t') \tag{4}$$

We shall work here assuming that Eqs. (1) are to be interpreted in the Stratonovic sense and consequently we are allowed to use the normal rules of calculus, as we shall do. This is no restriction, since if another interpretation is used,[4,5] we can always rewrite the equations in the Stratonovic sense, adding supplementary terms (this can be done, for instance, using the techniques in Ref. 6).

In order to have a cusp bifurcation as a codimension 1 situation in (1), we assume the symmetry $U_1 \to -U_1$ in the absence of noise. This symmetry plays no special role in the method, which works in its absence, and we only impose it to have a leading cubic term (i.e., a cusp bifurcation) in the normal form, which guarantees the existence of the stationary probability if its coefficient is negative. We are looking for the asymptotic behavior of $\mathbf{U}$ for times $t \gg \sup |\gamma_\alpha|^{-1}$, $\alpha \geqslant 2$, and to obtain it we make the ansatz (which is proved in Refs. 2 and 3) that $\mathbf{U}$ can be asymptotically expressed in terms of a critical variable C in the form $(U_\alpha^{[1]} = \delta_{\alpha_1}$, $\eta = \varepsilon^{1/2})$

$$\mathbf{U} = \sum_{r\geqslant 1} C^r \sum_{\alpha=1}^{N} U_\alpha^{[r]}\mathbf{e}_\alpha + \eta \sum_{r\geqslant 0} C^r \sum_{\alpha=1}^{N} V_\alpha^{[r]}(t)\,\mathbf{e}_\alpha \tag{5}$$

and that C obeys an equation of the form

$$\partial_t C = \sum_{r\geqslant 2} f^{[r]} C^r + \eta \sum_{r\geqslant 0} g^{[r]}(t)\, C^r \tag{6}$$

where $\{U_\alpha^{[r]}, f^{[r]}\}$ are constants to be determined and $\{V_\alpha^{[r]}(t), g^{[r]}(t)\}$ are stochastic processes to be determined. One obtains equations for these quantities by direct substitution of (5) and (6) in (1) and by identification of the left- and right-hand sides of (1) at each order $[j, r]$, where j is the order in η (0 or 1) and r is the order in C. In doing this, one calculates the left-hand side of (1) as

$$\partial_t \mathbf{U} = \partial_t C \frac{\partial \mathbf{U}}{\partial C} + \hat{\partial}_t \mathbf{U} \tag{7}$$

where $\hat{\partial}_t$ stands for derivatives with respect to the explicit t dependence in (5), which is contained in the $V_\alpha^{[r]}(t)$. We have at order $[0, r]$ an equation of the form

$$-\gamma_\alpha U_\alpha^{[r]} = I_\alpha^{[r]} - f^{[r]}\delta_{\alpha 1}, \qquad 1 \leqslant \alpha \leqslant N \tag{8}$$

where $I_\alpha^{[r]}$ is a quantity constructed in terms of $\{U_\alpha^{[s]}, f^{[s]}, s<r\}$, which means that we can try to solve (8) by recursion in r using $U_\alpha^{[1]}=\delta_{\alpha 1}$ and $f^{[1]}=0$. The symmetry $U_1 \to -U_1$ implies:

(a) $I_1^{[2n]}=0$, $n \geqslant 1$ (then we take $f^{[2n]}=0$, $n \geqslant 1$, $U_1^{[2n]}=0$) and $I_\alpha^{[2n]} \neq 0$, $\alpha \geqslant 2$, which gives $U_\alpha^{[2n]}=-(\gamma_\alpha)^{-1} I_\alpha^{[2n]}$.

(b) $I_1^{[2n+1]} \neq 0$ (then we take $f^{[2n+1]}=I_1^{[2n+1]}$, $U_1^{[2n+1]}=0$, $n \geqslant 1$) and $I_\alpha^{[2n+1]}=0$, $\alpha \geqslant 2$, which gives $U_\alpha^{[2n+1]}=0$. We know now the parts independent of η (i.e., in the absence of noise) of (5) and (6), which are

$$\partial_t C=\sum_{n \geqslant 1} f^{[2n+1]} C^{2n+1}+O(\eta) \tag{9}$$

$$\mathbf{U}=C\mathbf{e}_1+\sum_{n \geqslant 1} C^{2n} \sum_{\alpha \geqslant 2} U_\alpha^{[2n]} \mathbf{e}_\alpha+O(\eta) \tag{10}$$

Let us look now at the noise-dependent terms in (1). At order $[1,0]$ one obtains

$$(\partial_t-\gamma_\alpha)\, V_\alpha^{[0]}(t)=D_\alpha(t)-g^{[0]}(t)\, \delta_{\alpha 1} \tag{11}$$

Since $\gamma_1=0$, we take $g^{[0]}(t)=D_1(t)$, $V_1^{[0]}(t)=0$, and for $\alpha \geqslant 2$ we solve (11) to obtain

$$V_\alpha^{[0]}(t)=e^{\gamma_\alpha t} \int_0^t dt'\, e^{-\gamma_\alpha t'} D_\alpha(t'), \qquad \alpha \geqslant 2 \tag{12}$$

At order $[1, r]$ we shall have an equation of the form

$$(\partial_t-\gamma_\alpha)\, V_\alpha^{[r]}(t)=J_\alpha^{[r]}(t)-g^{[r]}(t)\, \delta_{\alpha 1} \tag{13}$$

where $J_\alpha^{[r]}(t)$ is constructed with $\{U_\alpha^{[s]}, s \leqslant r+1; f^{[s]}, s \leqslant r; V_\alpha^{[s]}(t), s<r; g^{[s]}(t), s<r\}$ and we see that we can again solve (13) by recursion in r ($U_\alpha^{[s]}$ and $f^{[s]}$ are already known from the noise-independent calculation). In summary, we have that (5) and (4) are of the form (we write V_α for $V_\alpha^{[0]}$)

$$U_1=C \tag{14}$$

$$U_\alpha=F_\alpha(C)+\eta V_\alpha(t)+O(\eta C) \tag{15}$$

$$\partial_t C=-\lambda C^3+\eta D_1(t)+O(C^5, \eta C) \tag{16}$$

$$(\partial_t-\gamma_\alpha)\, V_\alpha(t)=D_\alpha(t), \qquad \alpha \geqslant 2 \tag{17}$$

Here $U_\alpha=F_\alpha(C)=\rho_\alpha C^2+O(C^4)$ is obtained from (10) and is just the equation of the center manifold $U_\alpha = F_\alpha(U_1)$, $\alpha \geqslant 2$.[7] In terms of the original quantities in (2), one has

$$\rho_\alpha = -(\gamma_\alpha)^{-1} u^{(2)}_{\alpha;1,1}, \qquad \alpha \geqslant 2$$
$$\lambda = -\left[u^{(3)}_{1;111} - 2 \sum_{\rho=2}^{N} (\gamma_\rho)^{-1} u^{(2)}_{1;\rho 1} u^{(2)}_{\rho;11} \right] \tag{19}$$

Equation (16) is now the (stochastic) normal form of our initial system (1) at the critical point. In the neighborhood of the instability the form of (16) will be $\partial_t C = \mu C - \lambda C^3 + \eta D_1(t)$, where μ is an unfolding parameter. The stationary probability for C will exist if $\lambda > 0$ and is given by

$$p_{\mathrm{st}}(C) = \frac{2s}{\varepsilon^{1/4}} \frac{1}{\Gamma(1/4)} \exp\left(-\frac{s^4}{\varepsilon} C^4 \right); \qquad s^4 = \frac{\lambda}{2Q_{11}} \tag{20}$$

This expression shows that $C^4 = O(\varepsilon)$, i.e., if we put $C = \varepsilon^{1/4} c$, then

$$\bar{p}_{\mathrm{st}}(c) = \varepsilon^{1/4} p_{\mathrm{st}}(C) = \frac{2s}{\Gamma(1/4)} \exp(-s^4 c^4) \tag{21}$$

is independent of ε. On the other hand, from (17) we obtain the time-independent probability in the stationary state

$$p'_{\mathrm{st}}(V_\alpha) = \frac{r_\alpha}{\sqrt{\pi}} \exp[-r_\alpha^2 V_\alpha^2]; \qquad r_\alpha^2 = -\frac{\gamma_\alpha}{Q_{\alpha\alpha}} \tag{22}$$

Using (15), now we can obtain the conditional probability $p_{\mathrm{st}}(U_\alpha \mid C)$ as

$$p_{\mathrm{st}}(U_\alpha \mid C) = \int dV_\alpha \, p'_{\mathrm{st}}(V_\alpha) \, \delta(U_\alpha - F_\alpha(C) - \eta V_\alpha) \tag{23}$$

which gives $(2 \leqslant \alpha \leqslant N)$

$$p_{\mathrm{st}}(U_\alpha \mid C) = \frac{r_\alpha}{(\varepsilon\pi)^{1/2}} \exp\left[-r_\alpha^2 \frac{[U_\alpha - F_\alpha(C)]^2}{\varepsilon} \right] \tag{24}$$

This shows that the distribution of the variables U_α around the center manifold $U_\alpha = F_\alpha(C)$ is Gaussian. This is in fact a very general result, which is not a consequence of the approximation we took for (15) and we have proved[3] it to any order $O(C^n)$ [at higher orders the width of the Gaussian in (24) becomes C dependent, a fact used as an assumption in Ref. 8].

From (23) we see now that the marginal probability is given by $[F_\alpha(C) = \rho_\alpha C^2 + O(C^4)]$

$$p_{\rm st}(U_\alpha) = \int dC\, dV_\alpha\, p'_{\rm st}(V_\alpha)\, \delta(U_\alpha - \rho_\alpha C^2 - \varepsilon^{1/2} V_\alpha)\, p_{\rm st}(C) \tag{25}$$

with $p_{\rm st}(C)$ given by (20). Since the scaling of C is $C = \varepsilon^{1/4} c$ [see (21)], we see in (25) that the appropriate scaling for U_α is

$$U_\alpha = \varepsilon^{1/2} u_\alpha = \varepsilon^{1/2}(\rho_\alpha c^2 + V_\alpha) \tag{26}$$

Then $\bar{p}_{\rm st}(u_\alpha) = \varepsilon^{1/2} p_{\rm st}(U_\alpha)$ is independent of ε and is given by

$$\bar{p}_{\rm st}(u_\alpha) = \int dc\, dV_\alpha\, p'_{\rm st}(V_\alpha)\, \delta(u_\alpha - \rho_\alpha c^2 - V_\alpha)\, \bar{p}_{\rm st}(c) \tag{27}$$

which is explicitly [from (21) and (22)]

$$\bar{p}_{\rm st}(u_\alpha) = \frac{2 r_\alpha s_\alpha}{\sqrt{\pi}} \frac{1}{\Gamma(1/4)} \int_{-\infty}^{\infty} dv \exp[-r_\alpha^2 (u_\alpha - v^2)^2 - s_\alpha^4 v^4] \tag{28}$$

where we put $s_\alpha = s\ (\rho_\alpha)^{-1/2}$, taking $\rho_\alpha > 0$ for simplicity. This integral has the value

$$\bar{p}_{\rm st}(u_\alpha) = (-u_\alpha)^{1/2} K_{1/4}(\xi_\alpha u_\alpha^2) \exp(-\zeta_\alpha u_\alpha^2); \qquad u_\alpha < 0 \tag{29}$$

$$\bar{p}_{\rm st}(u_\alpha) = u_\alpha^{1/2} [K_{1/4}(\xi_\alpha u_\alpha^2) + \sqrt{2}\, \pi I_{1/4}(\xi_\alpha u_\alpha^2)] \exp(-\zeta_\alpha^2 u_\alpha^2); \qquad u_\alpha > 0 \tag{30}$$

with

$$\xi_\alpha = \frac{r_\alpha^4}{2(s_\alpha^4 + r_\alpha^2)}, \qquad \zeta_\alpha = \frac{r_\alpha^2 (2 s_\alpha^4 + r_\alpha^2)}{2(s_\alpha^4 + r_\alpha^2)}$$

Here $K_{1/4}$ and $I_{1/4}$ are Bessel functions.[9,10]

The case we have been discussing is the generic one and we see that here U_α has a Gaussian scaling $\varepsilon^{1/2}$ [see (27) and (28)]. However, the explicit expression for $\bar{p}_{\rm st}(u_\alpha)$ is not strictly a Gaussian, as can be seen in (29) and (30). One has two kinds of nongeneric cases appearing as codimension 2 situations: (a) $\lambda = 0$, and (b) $\rho_\alpha \neq 0$ for some $\alpha \geqslant 2$.

In case (a), Eq. (16) becomes

$$\partial_t C = -\lambda' C^5 + \eta D_1(t) + O(C^7, \eta C) \tag{31}$$

with stationary solution (for $\lambda' > 0$)

$$p_{st}(C) = \frac{A}{\varepsilon^{1/6}} \exp\left(-\frac{\lambda' C^6}{3\varepsilon Q_{11}}\right)$$
$$A^{-1} = \int dC \exp\left(-\frac{\lambda' C^6}{3Q_{11}}\right) = \frac{\Gamma(\frac{1}{6})}{6}\left(\frac{3Q_{11}}{\lambda'}\right)^{1/6} \tag{32}$$

The scaling is now $C = \varepsilon^{1/6} c$, which makes

$$\bar{p}_{st}(c) = \varepsilon^{1/6} p_{st}(C) = A \exp\left(-\frac{\lambda' c^6}{3Q_{11}}\right) \tag{33}$$

independent of ε. Equation (25) becomes

$$p_{st}(U_\alpha) = A \int dc\, dV_\alpha\, p'_{st}(V_\alpha)\, \delta(U_\alpha - \varepsilon^{1/3} \rho_\alpha c^2 - \varepsilon^{1/2} V_\alpha) \exp\left(-\frac{\lambda' c^6}{3Q_{11}}\right) \tag{34}$$

We see now that the appropriate scaling for U_α is $U_\alpha = \varepsilon^{1/3} u_\alpha$, which makes the argument of the δ-function in (34) become

$$\varepsilon^{1/3}(u_\alpha - \rho_\alpha c^2 - \varepsilon^{1/6} V_\alpha) = \varepsilon^{1/3}(u_\alpha - \rho_\alpha c^2)$$

in the $\varepsilon \to 0$ limit. Then (34) gives for $\bar{p}_{st}(u_\alpha) = \varepsilon^{1/3} p_{st}(U_\alpha)$ the ε-independent expression [using $\int dV_\alpha\, p'_{st}(V_\alpha) = 1$]

$$\bar{p}_{st}(u_\alpha) = A \int dc\, \delta(u_\alpha - \rho_\alpha c^2) \exp\left(-\frac{\lambda' c^6}{3Q_{11}}\right) \tag{35}$$

This gives for $\rho_\alpha > 0$ that $\bar{p}_{st}(u_\alpha) = 0$ for $u_\alpha < 0$, and for $u_\alpha > 0$ one obtains

$$\bar{p}_{st}(u_\alpha) = \frac{A}{(\rho_\alpha u_\alpha)^{1/2}} \exp\left(-\frac{\lambda' u_\alpha^3}{3\rho_\alpha^2 Q_{11}}\right) \tag{36}$$

which corresponds indeed to the non-Gaussian scaling $U_\alpha = \varepsilon^{1/3} u_\alpha$.

In the second nongeneric case (b) one has $\rho_\alpha = 0$ and the equation for the center manifold $U_\alpha = F_\alpha(C) = \rho'_\alpha C^4 + O(C^6)$ starts at order four. Equation (25) is now replaced by

$$p_{st}(U_\alpha) = \int dC\, dV_\alpha\, p'_{st}(V_\alpha)\, \delta(U_\alpha - \rho'_\alpha C^4 - \varepsilon^{1/2} V_\alpha)\, p_{st}(C) \tag{37}$$

with $p_{st}(C)$ given again by (20), which shows that the scaling for C is $C = \varepsilon^{1/4} c$, which causes the argument of the δ-function to become

$U_\alpha - \varepsilon \rho'_\alpha c^4 - \varepsilon^{1/2} V_\alpha = U_\alpha - \varepsilon^{1/2} V_\alpha$ in the $\varepsilon \to 0$ limit. Then (37) reduces to [using $\int dC\, p_{\mathrm{st}}(C) = 1$ with $p_{\mathrm{st}}(C)$ given here by (21)]

$$p_{\mathrm{st}}(U_\alpha) = \int dV_\alpha\, p'_{\mathrm{st}}(V_\alpha)\, \delta(U_\alpha - \varepsilon^{1/2} V_\alpha) \tag{38}$$

In this situation U_α is decoupled from $U_1 = C$, and from (38) we see that its scaling is Gaussian, $U_\alpha = \varepsilon^{1/2} u_\alpha$, which gives for $\bar{p}_{\mathrm{st}}(u_\alpha) = \varepsilon^{1/2} p_{\mathrm{st}}(U_\alpha)$ the Gaussian probability

$$\bar{p}_{\mathrm{st}}(u_\alpha) = p'_{\mathrm{st}}(u_\alpha) = \frac{r_\alpha}{\sqrt{\pi}} \exp(-r_\alpha^2 u_\alpha^2) \tag{39}$$

We finally remark that at the dominant order in the weak noise limit that we have been considering the multiplicative noise in (1) plays no role.

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